ME-221
SOLUTIONS FOR PROBLEM SET 10

Problem 1

Given the system represented by the transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10+s}{2s^2+4s+1}$$

a) The steady-state (DC) gain is given by:

$$K = \lim_{s\to 0} G(s) = 10$$

b) The amplitude ratio and the phase shift are calculated as:

$$|G(j\omega)| = \frac{|10+j\omega|}{|(1-2\omega^2)+4j\omega|} = \frac{\sqrt{100+\omega^2}}{\sqrt{(1-2\omega^2)^2+16\omega^2}}$$

$$arg[G(j\omega)] = arg[10+j\omega] - arg[(1-2\omega^2)+4j\omega]$$

$$= arctan(\frac{\omega}{10}) - [\pi + arctan(\frac{4\omega}{1-2\omega^2})$$

Input is given by $u(t) = 5sin(\pi t)$. Thus, $\omega = \pi = 3.14$ [rad/s]

$|G(j\pi)| = 0.465$ and

$arg[G(j\pi)] = -2.246$ [rad]

The steady-state output is $y_{ss}(t) = 5|G(jw)|sin(\pi t + arg[G(j\omega)]) = 2.325sin(\pi t - 2.246)$

Problem 2

a) We start with converting the transfer function into the standard form:

$$G(s) = \frac{12}{(s+4)(5s+1)} = \frac{3}{(0.25s+1)(5s+1)}$$

The magnitude and phase are given by:

$$|G(jw)| = \frac{3}{(\sqrt{\frac{\omega^2}{16}+1})(\sqrt{25\omega^2+1})}$$
$$arg[G(jw)] = 0 - arg[1+j0.25\omega] - arg[1+j5\omega]$$
$$= -arctan(\frac{\omega}{4}) - arctan(5\omega)$$

Remember that the asymptotic analysis involves finding the magnitude and phase for frequencies at which i) $\omega << 0.2$ and $\omega << 4$, ii) $\omega >> 0.2$ and $\omega << 4$, and iii) $\omega >> 0.2$ and $\omega >> 4$.

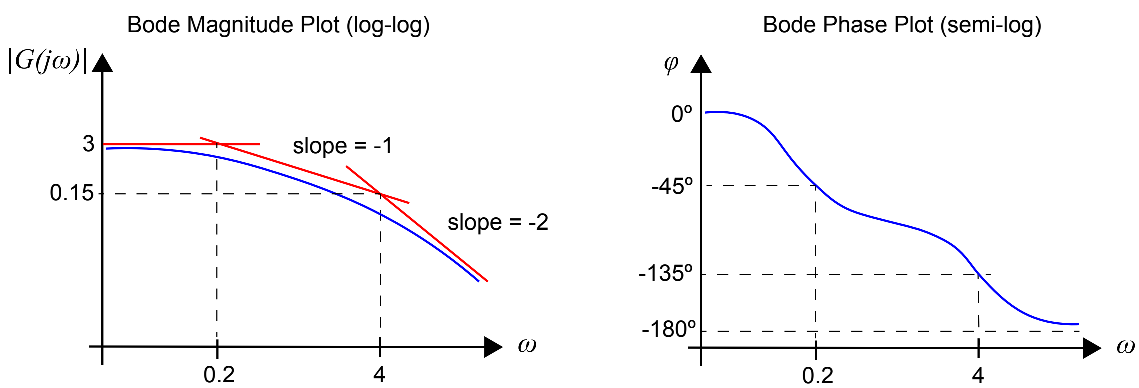

Figure 1: Bode plot for magnitude and phase. Red lines show the asymptotes and blue lines denote the actual curve.

b) The magnitude and phase can be easily found by entering the value of the frequency, which is given as $\omega = 2$. $|G(j2)| = 0.27$ and $\phi = arg[G(j2)] = -1.93$ rad.

Problem 3

The transfer function is given by $G(s) = 2\frac{bs+1}{0.5s+1}$. The transfer function is $G(s) = 2\frac{0.1s+1}{0.5s+1}$ for $b = 0.1$. The corresponding Bode plot is shown in Figure 2.

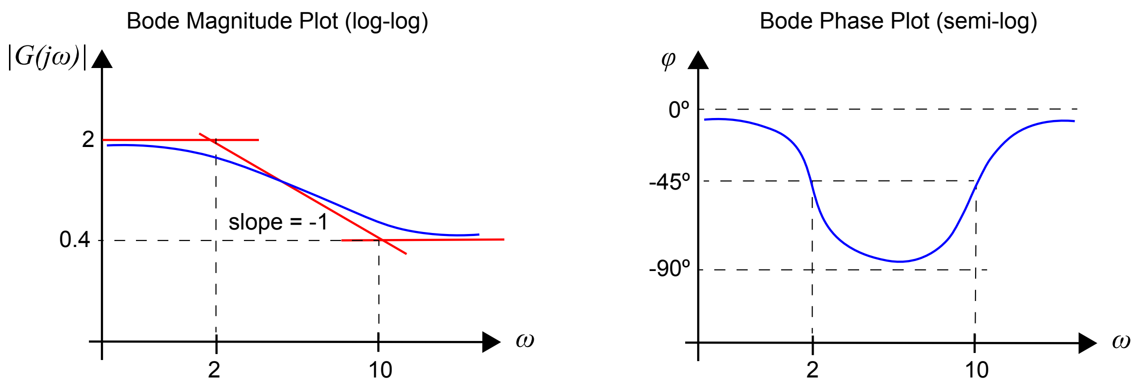

Figure 2: Bode plot for magnitude and phase. Red lines show the asymptotes and blue lines denote the actual curve.

Problem 4

The asymptotes of the bode diagram are shown in Figure 3. Note that the initial slope of the curve is -2 at low frequencies due to the $1/s^2$ term and at the corner frequency the slope of the curve changes from -2 to -3 due to the first order term in the denominator. Phase angle starts at $-180°$ at low frequencies due to the $1/s^2$ term and finally converges to $270°$ at high frequencies with the contribution coming from the first order term.

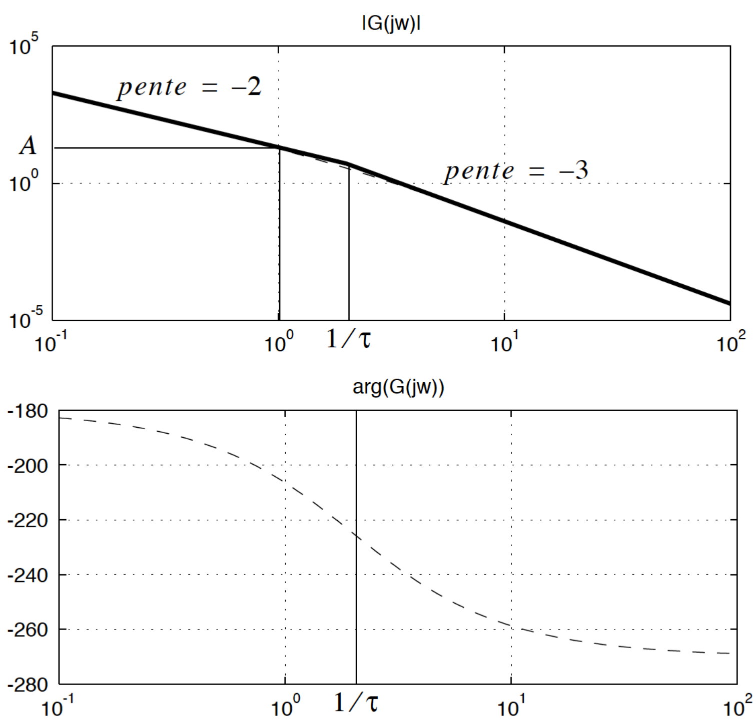

Figure 3: Bode diagram for magnitude and phase

b) The impulse response is, by definition, the inverse Laplace transform of the transfer function.

$$g(t) = \mathcal{L}^{-1}\big[G(s)\big] = \mathcal{L}^{-1}\big[\frac{A}{s^2(\tau s+1)}\big] = \mathcal{L}^{-1}\big[\frac{1}{s^2} + \frac{\tau^2}{\tau s+1} - \frac{\tau}{s}\big]$$
$$= A[t - \tau(1 - e^{-t/\tau}] \qquad for \quad t \geq 0$$

Problem 5

a) First component: $G_1(j\omega) = \frac{K}{s}$ where K must be determined

Second component: $G_2(j\omega) = (T_1 s + 1)$

Third component: $G_3(j\omega) = \frac{1}{T_2 s + 1}$

In order to determine K, one must consider the first two components at high frequencies ($s \to \infty$) which must result in A:

$$\lim_{s\to\infty} \frac{K(T_1 s + 1)}{s} = KT_1 = A \to K = \frac{A}{T_1}$$

The transfer function is obtained by multiplying these three components:

$$G(s) = \frac{A}{T_1}\frac{T_1 s + 1}{s(T_2 s + 1)}$$

b) We can then deduce the impulse response:

$$g(t) = \mathcal{L}^{-1}\big[G(s)\big] = \frac{A}{T_1}\big[1 + (\frac{T_1 - T_2}{T_2})e^{-(t/T_2)}\big] \qquad for \quad t \geq 0$$

c) The asymptotic approximation for the phase plot is shown in Figure 4.

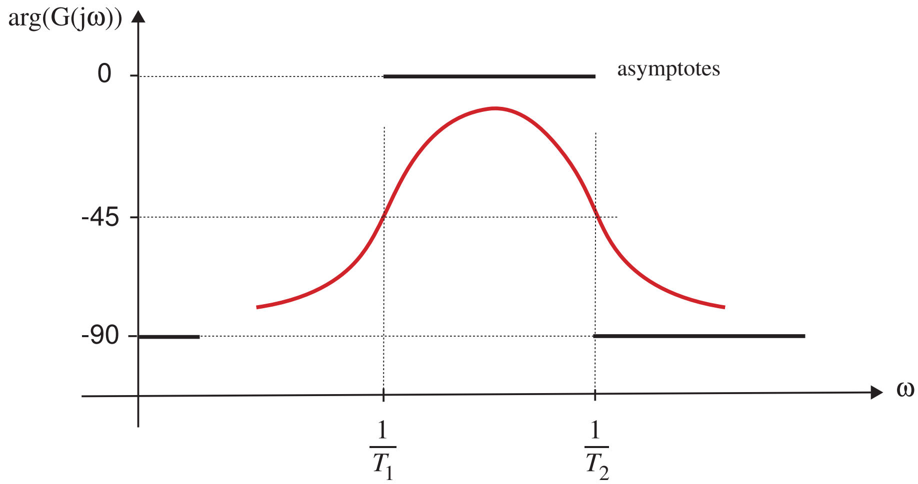

Figure 4: Asymptotes and Bode plot of the phase angle

Problem 6

The transfer function of the second order underdamped system is given by:

$$G(s) = \frac{1}{10s^2 + cs + 20} = \frac{1}{20}\frac{2}{s^2 + 0.1cs + 2}$$

The natural frequency is $\omega_0 = \sqrt{2}$, the gain is $K = 1/20$, and the damping coefficient is $\zeta = (0.1c)/(2\sqrt{2})$. Maximum steady state amplitude of y is reached at the resonant frequency ω_r and the magnitude of the sinusoidal transfer function at resonant frequency is given by

$$M_r = |G(j\omega_r)| = \frac{K}{2\zeta\sqrt{1-\zeta^2}} = \frac{1}{40\zeta\sqrt{1-\zeta^2}}$$

The amplitude of maximum steady state output is calculated by multiplying the amplitude of the input, which is given as 11, by the amplitude of the transfer function at the resonant frequency, M_r. We want to find the value of damping coefficient c that would make this value 3. That is to say, $M_r = 3/11$. The roots of the polynomial equation $\zeta^4 - \zeta^2 + (11/120)^2$ are 0.9958 and 0.092. As ζ is expected to be less than 0.707 to display resonance, $\zeta = 0.092$. For this value of ζ, $c = 20\sqrt{2}\zeta = 2.604$.